

# VECTOR ANALYSIS FOR LOCAL DIRICHLET FORMS AND QUASILINEAR PDE AND SPDE ON FRACTALS

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**ABSTRACT.** Starting with local regular symmetric Dirichlet forms, our paper studies some elements of vector analysis,  $L_p$ -spaces of vector fields and related Sobolev spaces. These tools are then employed to obtain existence and uniqueness results for some quasilinear elliptic PDE and some SPDE in variational form by classical methods. The setup is sufficiently general to be applied to Dirichlet forms on fractal spaces such as finitely ramified fractals and Sierpinski carpets.

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## 1. INTRODUCTION AND SETUP

The paper is concerned with some elements of vector analysis on topological spaces that carry a local regular Dirichlet form. We start from the notion of 1-forms based on energy as recently introduced by Cipriani and Sauvageot in [9, 10] and further studied in [23]. It is shown below that for local forms their concept may be seen as an extension of closely related and preceding constructions of Eberle, [13], based on abstract differential operators. This framework is then used to define  $L_p$ -spaces over fields of measurable Hilbert spaces, the space of 1-forms in the sense of [9, 10] appears for  $p = 2$ . Related Sobolev spaces come up naturally. Finally these tools are applied to quasilinear elliptic PDE in divergence and non-divergence form and finally to SPDE in variational form such as, for instance, the  $p$ -Laplace equation. We obtain existence and uniqueness results by classical fixed point and monotonicity arguments.

The main motivation for the present study comes from the analysis on fractals, cf. [25, 42]. For certain classes of fractal sets the existence of a Laplace operator has been proved, see [1, 2, 3, 17, 25, 31, 34, 40] and the references therein for some examples. Linear elliptic and parabolic PDE on fractals can then be treated by standard methods, [14]. Semilinear

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equations have been studied in [15]. There are also methods that apply to fully nonlinear problems, see for instance [4, 37] for porous medium equations. However, to our knowledge quasilinear equations of type  $\operatorname{div}(a(\nabla u)) = f$  or  $-\Delta u + b(\nabla u) = f$  with generally nonlinear coefficients  $a$  and  $b$  have not been considered so far, as a feasible notion of gradient  $\nabla$  on fractals had not yet been available.

In [9] and later in [10] and [23] a Hilbert space  $\mathcal{H}$  of 1-forms and a related analog  $\partial$  of the exterior derivation had been introduced by means of tensor products and energy norms, see Section 2 below for precise definitions. Its norm is most conveniently expressed in terms of energy measures in the sense of LeJan and Fukushima, [16, 30] (see also [38, 39]). Without too much effort a related notion of energy measures for 1-forms can be introduced, what yields a coherent picture and is useful in some applications.

The energy measure of a bounded energy finite function may be absolutely continuous with respect to the given reference measure or not. In Eberle [13, Section 3.2 and Appendix D] it is shown how to construct derivation operators if the energy measures are absolutely continuous for all functions from a dense algebra contained in the domain of the generator. On fractal spaces energy measures are typically singular with respect to the self-similar Hausdorff measure of the base space, cf. [5, 19, 20, 29]. However, the construction in [13, Theorem 3.11] is still possible if we choose a finite or countable pool of functions admitting energy densities and being energy dense in the space of bounded energy finite functions. Switching to a suitable measure if needed (called an energy dominant or Kusuoka measure, [22, 47]), this can be realized for any local regular Dirichlet form.

Following [13] we therefore obtain a measurable field of Hilbert spaces, [12, 46]. Rewriting the construction using some simple manipulations it can be shown that, roughly speaking, the resulting direct integral is a Hilbert space isomorphic to the space of 1-forms  $\mathcal{H}$ , and the direct integral of Eberle's fiberwise operators coincides with the derivation  $\partial$  in the sense of Cipriani and Sauvageot. Apart from minor modifications this material is not new in substance. However, the connection between these two constructions had not been established before. Our comparison shows that concerning their hypotheses, the construction in [9, 10] could be viewed as an extension of that in [13, Theorem 3.11], now based on a local regular Dirichlet form instead of an abstract differential operator. Even more importantly, our reasoning provides a constructive fiberwise interpretation for  $\mathcal{H}$  that carries over from [13] and is particularly useful to define  $L_p$ -spaces of vector fields. Our main motivation for this paper is, moreover, to establish a basis for further studies of first order differential operators on fractals, which have never carried out before. Examples of such operators and equations include the Navier-Stokes equations on fractals and Dirac operators, in particular operators related to magnetic fields.

By the self-duality of  $\mathcal{H}$  we regard its elements also as vector fields and  $\partial$  a gradient operator. As a first new result, a corresponding divergence operator is defined as the adjoint of  $\partial$ . Note that although Eberle considers the adjoint of the derivation operator, [13, Chapter 3 b), Section 1], in his case it is part of the basic hypotheses and the discussion there aims at constructing Sobolev spaces of functions rather than at investigating spaces of vector fields. Using the above mentioned fiberwise interpretation, it is then straightforward to define  $L_p$ -spaces of vector fields. To come up with Sobolev spaces of functions that make the derivation a closed operator also for  $p \neq 2$ , we roughly speaking assume that there exists a core  $\mathcal{C}_p$  of functions having  $p/2$ -integrable energy densities which is dense in  $L_p$  and moreover such that  $\mathcal{C}_p \otimes \mathcal{C}_p$  is dense in the corresponding  $L_p$ -space of vector fields. This is clearly satisfied in the

classical smooth context. To fulfill it for non-classical examples we propose to investigate (abstract) coordinates related to the Dirichlet form. Harmonic coordinates in the sense of [47] constitute a prototype case.

The final applications to PDE and SPDE then follow standard patterns that become applicable thanks to the definitions and results of the preceding sections.

Our basic setup is as follows:  $X$  is assumed to be a locally compact and second countable Hausdorff space;  $\mathcal{M}(X)$  denotes the space of (signed) Radon measures on  $X$  and  $\mathcal{M}_+(X)$  the cone consisting of its non-negative elements; a measure  $\mu \in \mathcal{M}(X)$  is an *admissible reference measure* on  $X$  if each open set  $U \subset X$  has positive measure  $\mu(U) > 0$ . In the sequel we assume that  $\mu$  is an admissible reference measure on  $X$  and, furthermore, we assume that  $(\mathcal{E}, \mathcal{F})$  is a local regular symmetric Dirichlet (energy) form on  $L_2(X, \mu)$ , cf. [16]. More exactly, we begin our arguments with an admissible reference measure  $\mu$ , and later switch to a different measure  $m$  if necessary, see Lemma 2.2 below.

Set  $\mathcal{C} := C_0(X) \cap \mathcal{F}$ . By regularity the space  $\mathcal{C}$  is dense in  $\mathcal{F}$ . It is an algebra, and endowed with the norm  $\|f\|_{\mathcal{C}} := \mathcal{E}(f)^{1/2} + \sup_X |f|$  we have in particular

$$(1) \quad \mathcal{E}(fg)^{1/2} \leq \|f\|_{\mathcal{C}} \|g\|_{\mathcal{C}},$$

as a consequence of the Markov property, see for instance [6]. Since  $C_0(X) \subset L_2(X, \mu)$  for any Radon measure  $\mu$ , we have

$$\mathcal{C} = \{f \in C_0(X) : \mathcal{E}(f) < \infty\}.$$

For any  $g, h \in \mathcal{C}$  there exists a unique signed finite measure  $\Gamma(g, h) \in \mathcal{M}(X)$  such that for any  $f \in \mathcal{C}$ ,

$$(2) \quad 2 \int_X f d\Gamma(g, h) = \mathcal{E}(fg, h) + \mathcal{E}(fh, g) - \mathcal{E}(gh, f).$$

Obviously  $\Gamma : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{M}(X)$  is a well defined symmetric bilinear mapping, and for any  $g \in \mathcal{C}$ ,  $\Gamma(g) \in \mathcal{M}_+(X)$ . Note that in particular  $\mathcal{E}(g, h) = \Gamma(g, h)(X)$ . Therefore  $\Gamma(f)$  is called the *energy measure of  $f$* , cf. [16, 30]. By approximation (2) may be extended to  $g, h \in \mathcal{F}$ .

In the next section the definition on the space  $\mathcal{H}$  of 1-forms is given and the concept of energy measure is extended to 1-forms. A fiberwise perspective is investigated and  $\mathcal{H}$  is shown to coincide with the direct integral considered in [13, Appendix D]. Section 3 introduces gradient and divergence, equipped with suitable domains. Sobolev spaces are introduced in Section 4, while Section 5 contains the applications to PDE and SPDE.

## 2. THE SPACE $\mathcal{H}$ AND WEIGHTED ENERGY MEASURES

By  $\mathcal{B}_b(X)$  we denote the space of bounded Borel functions on  $X$ . Consider  $\mathcal{C} \otimes \mathcal{B}_b(X)$ , endowed with the symmetric bilinear form

$$(3) \quad \langle a \otimes b, c \otimes d \rangle_{\mathcal{H}} := \int_X bd d\Gamma(a, c),$$

$a \otimes b, c \otimes d \in \mathcal{C} \otimes \mathcal{B}_b(X)$ , and let  $\|\cdot\|_{\mathcal{H}}$  denote the associated seminorm on  $\mathcal{C} \otimes \mathcal{B}_b(X)$ . We write

$$\ker \|\cdot\|_{\mathcal{H}} := \{a \otimes b \in \mathcal{C} \otimes \mathcal{B}_b(X) : \|a \otimes b\|_{\mathcal{H}} = 0\}.$$

The completion of  $\mathcal{C} \otimes \mathcal{B}_b(X) / \ker \|\cdot\|_{\mathcal{H}}$  with respect to  $\|\cdot\|_{\mathcal{H}}$  is denoted by  $\mathcal{H}$ . We refer to  $\mathcal{H}$  as the *space of differential 1-forms on  $X$* . Obviously it is a Hilbert space. Unlike for later

constructions we agree to use the same notation  $a \otimes b$  for a simple tensor from  $\mathcal{C} \otimes \mathcal{B}_b(X)$  and for its equivalence class in  $\mathcal{H}$ .

*Remark 2.1.* The space constructed from  $\mathcal{C} \otimes \mathcal{C}$  in an analogous manner agrees with  $\mathcal{H}$ .

The space  $\mathcal{H}$  becomes a bimodule if we declare the algebras  $\mathcal{C}$  and  $\mathcal{B}_b(X)$  to act on it in the following manner: For  $a \otimes b \in \mathcal{C} \otimes \mathcal{B}_b(X)$ ,  $c \in \mathcal{C}$  and  $d \in \mathcal{B}_b(X)$  set

$$(4) \quad c(a \otimes b) := (ca) \otimes b - c \otimes (ab)$$

and

$$(5) \quad (a \otimes b)d := a \otimes (bd).$$

As shown in [9] and [23], (4) and (5) extend to well defined left and right actions of the algebras  $\mathcal{C}$  and  $\mathcal{B}_b(X)$ , respectively. From (3) and the Leibniz rule for energy measures, see [16, Theorem 3.2.2], it can be seen that left and right multiplication agree for any  $c \in \mathcal{C}$ , and as

$$\max \{ \|(a \otimes b)c\|_{\mathcal{H}}, \|c(a \otimes b)\|_{\mathcal{H}} \} \leq \sup_X |c| \|a \otimes b\|_{\mathcal{H}},$$

it follows by approximation that they agree for all  $c \in \mathcal{B}_b(X)$ . See [23] for further details.

We continue the preceding ideas and develop a *global perspective*. The following results apply even if the energy measures are possibly not absolutely continuous with respect to the reference measure  $m$ . From  $\Gamma$  an  $\mathcal{M}(X)$ -valued bilinear mapping on  $\mathcal{H}$  can be constructed. It may be interpreted as a *weighted energy measure*. For simple tensors  $a \otimes b, c \otimes d \in \mathcal{H}$  set

$$(6) \quad \Gamma_{\mathcal{H}}(a \otimes b, c \otimes d) := bd \Gamma(a, c),$$

seen as an  $\mathcal{M}(X)$ -equality.

**Lemma 2.1.** *(6) extends to a well defined and uniquely determined symmetric bilinear mapping  $\Gamma_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{M}(X)$  such that for any  $\omega \in \mathcal{H}$ ,  $\Gamma_{\mathcal{H}}(\omega) \in \mathcal{M}_+(X)$ . For any  $\omega, \eta \in \mathcal{H}$  we have  $\Gamma_{\mathcal{H}}(\omega, \eta)(X) = \langle \omega, \eta \rangle_{\mathcal{H}}$ .*

*Proof.* First note that for any finite linear combination  $\sum_i a_i \otimes b_i \in \mathcal{C} \otimes \mathcal{B}_b(X)$  for which

$$\left\| \sum_i a_i \otimes b_i \right\|_{\mathcal{H}}^2 = \sum_i \sum_j b_i b_j \Gamma(a_i)(X) = 0,$$

we have

$$(7) \quad \sum_i \sum_j b_i b_j \Gamma(a_i) \equiv 0,$$

seen as equality in  $\mathcal{M}(X)$ . In fact, if  $c_1, \dots, c_m, d_1, \dots, d_n$  are such that

$$0 = \left\| \sum_{i=1}^m \sum_{j=1}^n c_i \otimes d_j \right\|_{\mathcal{H}}^2 = \int_X \left( \sum_{j=1}^n d_j \right)^2 d\Gamma \left( \sum_{i=1}^m c_i \right),$$

then  $\left( \sum_{j=1}^n d_j \right)^2 \Gamma \left( \sum_{i=1}^m c_i \right) \equiv 0$  in  $\mathcal{M}(X)$  because  $\Gamma \left( \sum_{i=1}^m c_i \right)$  is a non-negative measure. The case  $c_i = a_i$  and  $d_j = b_i$  if  $j = i$  and 0 otherwise yields (7). Now consider finite linear

combinations  $\sum_i f_i \otimes g_i \in \mathcal{H}$ . For each  $i$  let  $\tilde{f}_i \otimes \tilde{g}_i \in \mathcal{C} \otimes \mathcal{B}_b(X)$  be a representant of  $f_i \otimes g_i$  and set

$$(8) \quad \Gamma_{\mathcal{H}} \left( \sum_i f_i \otimes g_i \right) := \sum_i \sum_j \tilde{g}_i \tilde{g}_j \Gamma(\tilde{f}_i) \geq 0.$$

By the previous arguments (8) is a well defined element of  $\mathcal{M}(X)$ . Given a general 1-form  $\omega \in \mathcal{H}$ , let  $(\omega_k)_k$  be a sequence of finite linear combinations

$$\omega_k = \sum_{i=1}^{n_k} f_i^{(k)} \otimes g_i^{(k)} \in \mathcal{H}$$

approximating  $\omega$  in  $\mathcal{H}$ . For a non-negative function  $\varphi \in \mathcal{B}_b(X)$  obviously  $\sqrt{\varphi} \in \mathcal{B}_b(X)$  and by (5),

$$\begin{aligned} \lim_k \int_X \varphi d\Gamma_{\mathcal{H}}(\omega_k) &= \lim_k \sum_i \sum_j \int_X \varphi g_i^{(k)} g_j^{(k)} d\Gamma(f_i^{(k)}) \\ &= \lim_k \|\omega_k \sqrt{\varphi}\|_{\mathcal{H}}^2 \\ &= \|\omega \sqrt{\varphi}\|_{\mathcal{H}}^2. \end{aligned}$$

Set

$$(9) \quad \Gamma_{\mathcal{H}}(\omega)(\varphi) := \lim_k \int_X \varphi d\Gamma_{\mathcal{H}}(\omega_k).$$

For arbitrary  $\varphi \in \mathcal{B}_b(X)$  consider the standard decomposition  $\varphi = \varphi_+ - \varphi_-$  with  $\varphi_+ = \max(\varphi, 0)$ ,  $\varphi_- = \max(-\varphi, 0)$  and define a linear functional on  $\mathcal{B}_b(X)$  by

$$(10) \quad \Gamma_{\mathcal{H}}(\omega)(\varphi) := \lim_k \int_X \varphi d\Gamma_{\mathcal{H}}(\omega_k) = \lim_k \int_X \varphi_+ d\Gamma_{\mathcal{H}}(\omega_k) - \lim_k \int_X \varphi_- d\Gamma_{\mathcal{H}}(\omega_k).$$

As this equals  $\|\omega \sqrt{\varphi_+}\|_{\mathcal{H}}^2 - \|\omega \sqrt{\varphi_-}\|_{\mathcal{H}}^2$ , we have

$$(11) \quad |\Gamma_{\mathcal{H}}(\omega)(\varphi)| \leq 2 \sup_x |\varphi(x)| \|\omega\|_{\mathcal{H}}^2.$$

(10) and (11) hold in particular for any  $\varphi \in C_0(X)$ , (9) is non-negative if  $\varphi \geq 0$ . Hence by the Riesz representation theorem there exists a unique non-negative Radon measure  $\Gamma_{\mathcal{H}}(\omega)$  on  $X$  such that

$$\int_X \varphi d\Gamma_{\mathcal{H}}(\omega) = \Gamma_{\mathcal{H}}(\omega)(\varphi) \quad \text{for all } \varphi \in C_0(X).$$

By boundedness (11) and density this extends to all  $\varphi \in C_b(X)$ , and  $\Gamma_{\mathcal{H}}(\omega)$  is seen to be the weak limit of the measures  $\Gamma_{\mathcal{H}}(\omega_k)$ . Finally, a bilinear mapping  $\Gamma_{\mathcal{H}}$  is defined via polarization, and the last statement of the lemma follows easily from (8) and (9).  $\square$

To the support of the measure  $\Gamma_{\mathcal{H}}(\omega)$  we refer as the *support of the 1-form*  $\omega \in \mathcal{H}$ .

### Corollary 2.1.

- (i) If  $\omega \in \mathcal{H}$  is such that  $\|\omega\|_{\mathcal{H}} = 0$ , then  $\Gamma_{\mathcal{H}}(\omega) = 0$  in  $\mathcal{M}(X)$ .

(ii) For any  $\omega, \eta \in \mathcal{H}$  and any Borel set  $A$ ,

$$|\Gamma_{\mathcal{H}}(\omega, \eta)|(A) \leq \Gamma_{\mathcal{H}}(\omega)(A)^{1/2} \Gamma_{\mathcal{H}}(\eta)(A)^{1/2}$$

for any Borel set  $A \in \mathcal{B}(X)$ . In particular,  $\Gamma_{\mathcal{H}}(\omega, \eta) = 0$  in  $\mathcal{M}(X)$  if  $\omega$  and  $\eta$  have disjoint supports.

*Proof.* (i) is a consequence of (11). The first statement in (ii) follows by a standard argument, see e.g. [32, Proposition 3.3]: By Lemma 2.1,

$$0 \leq \Gamma_{\mathcal{H}}(\omega - \lambda\eta) = \Gamma_{\mathcal{H}}(\omega) - 2\lambda\Gamma_{\mathcal{H}}(\omega, \eta) + \lambda^2\Gamma_{\mathcal{H}}(\eta).$$

For any relatively compact Borel set  $A$  and any  $\lambda > 0$ ,

$$|\Gamma_{\mathcal{H}}(\omega, \eta)|(A) \leq \frac{1}{2} (\lambda^{-1}\Gamma_{\mathcal{H}}(\omega)(A) + \lambda\Gamma_{\mathcal{H}}(\eta)(A)).$$

If, without loss of generality,  $\Gamma_{\mathcal{H}}(\eta) = 0$ , then we can let  $\lambda$  go to zero to see the left hand side is zero. If both  $\Gamma_{\mathcal{H}}(\omega)$  and  $\Gamma_{\mathcal{H}}(\eta)$  are nonzero, consider

$$\lambda = \frac{\Gamma_{\mathcal{H}}(\omega)(A)^{1/2}}{\Gamma_{\mathcal{H}}(\eta)(A)^{1/2}}$$

to arrive at the desired inequality. By the regularity properties of the measures it extends to arbitrary Borel sets. The last statement in (ii) is a simple consequence.  $\square$

The above picture can be complemented by a *fiberwise perspective*. The following fact is well known, see for instance [22, Lemmas 2.2-2.4]. For the convenience of the reader we briefly sketch it.

**Lemma 2.2.** *Given a regular Dirichlet form  $(\mathcal{E}, \mathcal{F})$  on  $L_2(X, \mu)$ , it is always possible to construct an admissible reference measure  $\tilde{m}$  such that for all  $f \in \mathcal{C}$ , the measure  $\Gamma(f)$  is absolutely continuous with respect to  $\tilde{m}$  and the density  $\frac{d\Gamma(f)}{d\tilde{m}}$  is in  $L_1(X, \tilde{m})$ .  $\tilde{m}$  may be chosen to be finite.*

As usual we write  $\mathcal{E}_1(f) := \mathcal{E}(f) + \|f\|_{L_2(X, \mu)}^2$ ,  $f \in \mathcal{F}$ .

*Proof.* As  $(\mathcal{F}, \mathcal{E}_1)$  is a separable Hilbert space, it possesses a dense subset  $\{e_n\}_n$  (in practice we may for instance take a countable orthonormal basis). For fixed  $n$ , let  $(\varphi_{n,k})_k$  be a sequence of functions from  $\mathcal{C}$  such that

$$\mathcal{E}_1(e_n - \varphi_{n,k})^{1/2} \leq 2^{-k}, \quad k \in \mathbb{N}.$$

Then  $\{\varphi_{n,k}\}_{k,n}$  is a countable family of functions from  $\mathcal{C}$ , clearly also dense in  $\mathcal{F}$  with respect to  $\mathcal{E}_1$ . Let  $\{f_n\}_n$  be an enumeration of this family. For all  $n \in \mathbb{N}$  consider the probability measure

$$m_n := \frac{\Gamma(f_n)}{\Gamma(f_n)(X)}$$

and put

$$\tilde{m} := \sum_{n=0}^{\infty} 2^{-n} m_n,$$

proceeding as in the proof of Lemma 2.1 the series is seen to converge in the weak topology. For any  $f \in \mathcal{C}$  there is some approximating sequence  $(f_{n_j})_j$  and by construction each  $\Gamma(f_{n_j})$

is absolutely continuous with respect to  $\tilde{m}$ . If  $B \in \mathcal{B}(X)$  is such that  $\tilde{m}(B) = 0$ , then  $\Gamma(f_{n_j})(B) = 0$  for all  $j$  and since

$$|\Gamma(f)(B)^{1/2} - \Gamma(f_{n_j})(B)^{1/2}| \leq \Gamma(f - f_{n_j})(B)^{1/2} \leq \mathcal{E}(f - f_{n_j})^{1/2},$$

cf. [16, p.111], we have  $\Gamma(f)(B) = 0$ , too.

Indeed  $\tilde{m}$  is an admissible reference measure: Assume there were an open set  $U \subset X$  such that  $\Gamma(f_n)(U) = 0$  for all  $n$ . Let  $\varphi \in \mathcal{C}$  be nontrivial and supported in  $U$ . As  $\varphi$  can be approximated in the  $\mathcal{E}_1$ -norm by a sequence  $(f_{n_j})_j$ , this would entail

$$(12) \quad \mathcal{E}(\psi, \varphi) = 0 \text{ for all } \psi \in \mathcal{C} \text{ supported in } U,$$

recall for instance Corollary 2.1 (ii). But then  $\mathcal{E}(\varphi) = 0$ , a contradiction to the previous assumptions on  $\varphi$ . Consequently  $m_n(U) > 0$  for some  $n$ .  $\square$

Let us return to the fixed Dirichlet form  $(\mathcal{E}, \mathcal{F})$  on  $L_2(X, \mu)$  as used in (2) and (3). From now on we assume that

$m$  is an admissible reference measure such that for any  $f \in \mathcal{C}$ , the measure  $\Gamma(f)$

is absolutely continuous with respect to  $m$  and  $\Gamma(f) = \frac{d\Gamma(f)}{dm}$  is in  $L_1(X, m)$ .

If all energy measures  $\Gamma(f)$ ,  $f \in \mathcal{C}$ , are absolutely continuous with respect to  $\mu$ , we may use  $m := \mu$ . If not, we switch to the measure  $m := \tilde{m}$  from Lemma 2.2. As this is sufficient for later purposes, the above assumption is no additional restriction.

We recall a construction from [13]. Let  $\mathcal{A}_0 = \{f_n\}_n$  be a countable collection of functions which is  $\mathcal{E}$ -dense in  $\mathcal{C}$ , i.e. such that for any  $f \in \mathcal{C}$  there exists a sequence  $(f_{n_j})_j \subset \mathcal{A}_0$  with  $\lim_j \mathcal{E}(f - f_{n_j}) = 0$ . For any finite linear combination  $u = \sum_{i=1}^N \lambda_i f_i$  we have

$$\begin{aligned} 0 \leq \mathcal{E}(u, u) &= \sum_{i=1}^N \sum_{j=1}^N \lambda_i \lambda_j \int_X \Gamma_x(f_i, f_j) m(dx) \\ &= \int_X \bar{\lambda}^T (\Gamma_x(f_i, f_j))_{i,j=1,\dots,N} \bar{\lambda} m(dx), \end{aligned}$$

where  $\bar{\lambda} = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$  and  $\bar{\lambda}^T$  is its transpose. Choose Borel versions  $x \mapsto \Gamma_x(f_i, f_j)$  of the classes  $\Gamma(f_i, f_j) \in L_1(X, m)$  such that for all  $N \in \mathbb{N}$  and all  $x \in X$ , the matrix  $(\Gamma_x(f_i, f_j))_{i,j=1,\dots,N}$  is nonnegative definite over  $\mathbb{Q}^N$ . For two finite linear combinations  $u = \sum_i \lambda_i f_i$  and  $v = \sum_j \mu_j f_j$  from  $\mathcal{A} := \text{span}(\mathcal{A}_0)$  set

$$\Gamma_x(u, v) := \sum_i \sum_j \lambda_i \mu_j \Gamma_x(f_i, f_j).$$

Then for all  $x \in X$ ,  $\Gamma_x$  clearly is a non-negative definite bilinear form on  $\mathcal{A}$ . Consider the factor  $\mathcal{A}/\ker \Gamma_x$ , where  $\ker \Gamma_x := \{f \in \mathcal{A} : \Gamma_x(f) = 0\}$  and let  $d_x f$  denote the equivalence class of  $f \in \mathcal{A}$ . Define

$$(13) \quad (d_x f, d_x g)_{\mathcal{B}_x} = \Gamma_x(f, g)$$

for all  $f, g \in \mathcal{A}$  and let  $\mathcal{B}_x$  denote the completion of  $\mathcal{A}/\ker \Gamma_x$  in  $(\cdot, \cdot)_{\mathcal{B}_x}$ , clearly a Hilbert space. For convenience we recall the following definitions: A collection  $(H_x)_{x \in X}$  of Hilbert

spaces  $(H_x, (\cdot, \cdot)_{H_x})$  together with a subspace  $M$  of  $\prod_{x \in X} H_x$  is called a *measurable field of Hilbert spaces* if

- (i) an element  $\xi \in \prod_{x \in X} H_x$  is in  $M$  if and only if  $x \mapsto (\xi, \eta)_{H_x}$  is measurable for any  $\eta \in M$ ,
- (ii) there exists a countable set  $\{\xi_i : i \in \mathbb{N}\} \subset M$  such that for all  $x \in X$  the span of  $\{\xi_i(x) : i \in \mathbb{N}\}$  is dense in  $H_x$ .

The elements of  $M$  are usually referred to as *measurable sections*. Two measurable fields of Hilbert spaces  $(H_x)_{x \in X}$  and  $(\tilde{H}_x)_{x \in X}$  are *essentially isometric* if there are a null set  $\mathcal{N} \subset X$  and a collection  $(\Phi_x)_{x \in X \setminus \mathcal{N}}$  of isometries  $\Phi_x : H_x \rightarrow \tilde{H}_x$  such that  $\xi \in \prod_{x \in X} H_x$  is a member of  $\mathcal{M}$  if and only if  $x \mapsto \Phi_x(\xi(x)) \in \tilde{\mathcal{M}}$ . If  $\mathcal{N}$  may be chosen to be empty, we say that  $(H_x)_{x \in X}$  and  $(\tilde{H}_x)_{x \in X}$  are *isometric*.

*Remark 2.2.* Orthonormalizing the  $\xi_i$  from (ii) in the respective spaces one obtains the following useful fact: There is a countable set  $\{\eta_i : i \in \mathbb{N}\} \subset M$  such that for any  $x$  with  $H_x$  infinite-dimensional, it provides a orthonormal basis and for any  $x$  with  $\dim H_x = d(x)$ ,  $\eta_1(x), \dots, \eta_{d(x)}(x)$  is an orthonormal basis and  $\eta_i(x) = 0$ ,  $i > d(x)$ . For a proof see [12, Proposition II.4.1] or [46, Lemma 8.12]. Note that every  $\eta_i(x)$  is a finite linear combination of elements  $\xi_j(x)$ .  $\{\eta_i : i \in \mathbb{N}\} \subset M$  is then referred to as a measurable field of orthogonal bases.

**Lemma 2.3.**

- (i)  $(\mathcal{B}_x)_{x \in X}$  is a measurable field of Hilbert spaces.
- (ii) Different choices of versions above lead to essentially isometric fields of Hilbert spaces.

*Proof.* Let  $\mathcal{M}$  be the subspace of all  $\xi \in \prod_{x \in X} \mathcal{B}_x$  such that  $x \mapsto (\xi(x), d_x f_n)_{\mathcal{B}_x}$  is measurable for any  $n$ . Obviously all  $x \mapsto d_x f$ ,  $f \in \mathcal{A}$ , are in  $\mathcal{M}$ . For general  $\xi \in \prod_{x \in X} \mathcal{B}_x$  and each  $x \in X$  there is a sequence  $(g_k) \subset \mathcal{A}$  such that

$$\lim_k \|\xi(x) - d_x g_k\|_{\mathcal{B}_x} = 0.$$

Hence a section  $\xi$  is in  $\mathcal{M}$  if and only if  $x \mapsto (\xi(x), d_x f_n)_{\mathcal{B}_x}$  are measurable for all  $n \in \mathbb{N}$ . This shows (i).

To see (ii), assume  $x \mapsto \tilde{\Gamma}_x(f_i, f_j)$  are further versions of  $\Gamma(f_i, f_j) \in L_1(X, m)$  so that the previous agreements are valid and denote the similarly constructed spaces by  $\tilde{\mathcal{B}}_x$ . Then there exists a null set  $\mathcal{N}$  such that

$$(\tilde{d}_x f_i, \tilde{d}_x f_j)_{\tilde{\mathcal{B}}_x} = (d_x f_i, d_x f_j)_{\mathcal{B}_x}$$

for all  $i, j \in \mathbb{N}$  and  $x \in X \setminus \mathcal{N}$ . By the density of  $\mathcal{A}/\ker \Gamma_x$  in  $\mathcal{B}_x$  and  $\mathcal{A}/\ker \tilde{\Gamma}_x$  in  $\tilde{\mathcal{B}}_x$  we obtain a unique isometry  $\Phi_x$  from  $\mathcal{B}_x$  onto  $\tilde{\mathcal{B}}_x$  for any  $x \in X \setminus \mathcal{N}$ . If now  $\xi \in \mathcal{M}$  then

$$(\Phi_x(\xi(x)), \tilde{d}_x f_n)_{\tilde{\mathcal{B}}_x} = (\xi(x), d_x f_n)_{\mathcal{B}_x}$$

for  $x \in X \setminus \mathcal{N}$  and all  $n \in \mathbb{N}$ , and the right-hand side is a measurable function of  $x$ . Therefore  $\Phi_x(\xi(x))$  is a measurable section. Similarly for the converse direction.  $\square$

This construction may be rephrased as follows. For any point  $x \in X$  and arbitrary simple tensors  $a \otimes b, c \otimes d \in \mathcal{A} \otimes \mathcal{B}_b(X)$  put

$$(14) \quad \Gamma_{\mathcal{H}, x}(a \otimes b, c \otimes d) := b(x)d(x)\Gamma_x(a, c).$$



As a consequence of the above choice of versions every  $\Gamma_{\mathcal{H},x}$ ,  $x \in X$ , defines a non-negative definite bilinear form on  $\mathcal{A} \otimes \mathcal{B}_b(X)$ . Set

$$\ker \Gamma_{\mathcal{H},x} := \left\{ \sum_i a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{B}_b(X) : \Gamma_{\mathcal{H},x}(\sum_i a_i \otimes b_i) = 0 \right\}$$

and let  $\mathcal{H}_x$  be the Hilbert space obtained as the completion of  $\mathcal{A} \otimes \mathcal{B}_b(X)/\ker \Gamma_{\mathcal{H},x}$  with respect to scalar product determined by

$$([a \otimes b]_x, [c \otimes d]_x)_{\mathcal{H}_x} = \Gamma_{\mathcal{H},x}(a \otimes b, c \otimes d),$$

where  $[a \otimes b]_x \in \mathcal{A} \otimes \mathcal{B}_b(X)/\ker \Gamma_{\mathcal{H},x}$  denotes the equivalence class of  $a \otimes b$ . Note that

$$(15) \quad [a \otimes b]_x = [a \otimes b(x)]_x = b(x)[a \otimes \mathbf{1}]_x \text{ for any } x \in X$$

and any  $a \otimes b \in \mathcal{A} \otimes \mathcal{B}_b(X)$ , because  $\Gamma_{\mathcal{H},x}(a \otimes (b - b(x))) = 0$  by (14).

**Lemma 2.4.**  $(\mathcal{H}_x)_{x \in X}$  is a measurable field of Hilbert spaces on  $X$ .  $(\mathcal{H}_x)_{x \in X}$  and  $(\mathcal{B}_x)_{x \in X}$  are isometric.

*Proof.* The first assertion may be seen as in the previous lemma. For any  $x \in X$  define a bilinear mapping  $\Psi_x : \mathcal{A}/\ker \Gamma_x \rightarrow \mathcal{H}_x$  by

$$(16) \quad \Psi_x(d_x a) := [a \otimes \mathbf{1}]_x, \quad a \in \mathcal{A}.$$

Since

$$(17) \quad \|\Psi_x(d_x a)\|_{\mathcal{H}_x}^2 = \|[a \otimes \mathbf{1}]_x\|_{\mathcal{H}_x}^2 = \Gamma_{\mathcal{H},x}(a \otimes \mathbf{1}) = \Gamma_x(a) = \|d_x a\|_{\mathcal{B}_x}^2$$

and  $d_x \tilde{a} = d_x a$  if and only if  $\Gamma_x(\tilde{a} - a) = 0$ ,  $\Psi_x$  is well defined. By (17) and denseness it extends to a uniquely determined isometry from  $\mathcal{B}_x$  into  $\mathcal{H}_x$ .  $\Psi_x$  is also surjective: For any  $[a \otimes b]_x \in \mathcal{A} \otimes \mathcal{B}_b(X)/\ker \Gamma_{\mathcal{H},x}$  consider  $b(x)d_x a$ . Then by linearity and (15),  $\Psi_x(b(x)d_x a) = b(x)[a \otimes \mathbf{1}]_x = [a \otimes b]_x$ . On the other hand,  $\mathcal{A} \otimes \mathcal{B}_b(X)/\ker \Gamma_{\mathcal{H},x}$  is dense in  $\mathcal{H}_x$ .  $\square$

**Lemma 2.5.**  $\mathcal{A} \otimes \mathcal{B}_b(X)$  is dense in  $\mathcal{H}$ .

*Proof.* By construction, any simple tensor  $a \otimes b \in \mathcal{C} \otimes \mathcal{B}_b(X)$  can be approximated by elements of  $\mathcal{A} \otimes \mathcal{B}_b(X)$ .  $\square$

Recall that given a measurable field of Hilbert spaces  $(H_x)_{x \in X}$ , a measurable section  $\xi$  is called *square-integrable* if

$$(18) \quad \int_X \|\xi(x)\|_{H_x}^2 m(dx) < \infty.$$

The set of all square-integrable sections together with the scalar product induced by (18) is called the *direct integral* of  $(H_x)_{x \in X}$  and denoted by  $\int_X^\oplus H_x m(dx)$ .

*Remark 2.3.* If  $\{\eta_i : i \in \mathbb{N}\}$  is a measurable field of orthonormal bases according to Remark 2.2 and  $H = \omega \in \int_X^\oplus H_x m(dx)$ , then the sections  $\omega_n$ , given by

$$\omega_n(x) = \sum_{i=0}^n (\omega(x), \eta_i(x))_{H_x} \eta_i(x)$$

approximate  $\omega$  in  $H$ . A proof is given in [12, Proposition II.1.6].

Given  $a \otimes b \in \mathcal{A} \otimes \mathcal{B}_b$  with corresponding classes  $[a \otimes b]_x \in \mathcal{H}_x$ , the symbol  $[a \otimes b]$  denote the measurable section  $x \mapsto [a \otimes b]_x$ . Similarly for more general measurable sections  $\omega$ .

**Theorem 2.1.** *The Hilbert spaces  $\mathcal{H}$  and  $\int_X^\oplus \mathcal{H}_x m(dx)$  are isometric. In particular, for all  $\omega, \eta \in \mathcal{H}$ ,*

$$(\omega, \eta)_{\mathcal{H}} = \int_X^\oplus (\omega, \eta)_{\mathcal{H}_x} m(dx).$$

Consequently also  $\mathcal{H}$  and  $\int_X^\oplus \mathcal{B}_x m(dx)$  are isometric. In particular, up to isometry, the definition of 1-forms in [13, Chapter 3 b) and Appendix D] arises as a special case of that in [9, 10].

*Proof.* For simple tensors  $a \otimes b \in \mathcal{A} \otimes \mathcal{B}_b(X)$  set  $\chi(a \otimes b) := [a \otimes b]$  and extend linearly to a mapping  $\chi : \mathcal{A} \otimes \mathcal{B}_b(X) \rightarrow \int_X^\oplus (\omega, \eta)_{\mathcal{H}_x} m(dx)$ . Since

$$\begin{aligned} \int_X \| [a \otimes b]_x \|_{\mathcal{H}_x}^2 m(dx) &= \int_X b(x)^2 \| [a \otimes \mathbf{1}]_x \|_{\mathcal{H}_x}^2 m(dx) \\ &= \int_X b(x)^2 \Gamma_x(a) m(dx) = \| a \otimes b \|_{\mathcal{H}}^2, \end{aligned}$$

By denseness  $\chi$  extends to an isometry from  $\mathcal{H}$  into  $\int_X^\oplus \mathcal{H}_x m(dx)$ . To conclude surjectivity we make use of a totality argument from [13, Theorem 7.3.11]. Suppose  $\omega \in \int_X^\oplus \mathcal{H}_x m(dx)$  is such that

$$0 = (\omega, [a \otimes b])_{\mathcal{H}} = \int_X b(x) (\omega(x), [a \otimes \mathbf{1}]_x)_{\mathcal{H}_x} m(dx).$$

Then in particular  $(\omega(x), [a \otimes \mathbf{1}]_x)_{\mathcal{H}_x} = 0$  for all  $a \in \mathcal{A}_0$  for  $m$ -a.e.  $x$ . But finite linear combinations  $\sum_i \lambda_i [a_i \otimes \mathbf{1}]_x$  with functions  $a_i \in \mathcal{A}_0$  and rational coefficients  $\lambda_i$  are dense in the Hilbert space  $\mathcal{H}_x$ , therefore  $\omega(x) = 0$  for  $m$ -a.e.  $x$  and consequently  $\omega = 0$  in  $\int_X^\oplus \mathcal{H}_x m(dx)$ . Then the closure of the range  $Im \chi$  of  $\chi$  must be the entire direct integral.  $\square$

Let us agree upon the notation  $\Gamma_{\mathcal{H},x}(\omega, \eta) := (\omega, \eta)_{\mathcal{H}_x}$  for all  $\omega, \eta \in \mathcal{H}$ . Analogs of Lemma 2.1 and Corollary 2.1 now read as follows.

**Corollary 2.2.**

- (i) *The measure  $\Gamma_{\mathcal{H}}(\omega, \eta)$  from Lemma 2.1 is absolutely continuous with respect to  $m$ , and  $\Gamma_{\mathcal{H},\cdot}(\omega, \eta)$  is a version of the Radon-Nikodym density  $\frac{d\Gamma_{\mathcal{H}}(\omega, \eta)}{dm}$ .*
- (ii)  *$\Gamma_{\mathcal{H},\cdot}$  provides a well defined and uniquely determined bilinear mapping  $\Gamma_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \rightarrow L_1(X, m)$  such that for any  $\omega \in \mathcal{H}$ ,  $\Gamma_{\mathcal{H},\cdot}(\omega, \omega) \geq 0$   $m$ -a.e.*

*Proof.* (i) is obvious and (ii) is a simple consequence of Lemmas 2.3 and 2.4.  $\square$

**Corollary 2.3.**

- (i) *If  $\omega \in \mathcal{H}$  is such that  $\|\omega\|_{\mathcal{H}} = 0$ , then  $\Gamma_{\mathcal{H},\cdot}(\omega) = 0$  in  $L_1(X, m)$ .*
- (ii)  *$\Gamma_{\mathcal{H}}(\omega, \eta) = 0$  in  $L_1(X, m)$  if  $\omega$  and  $\eta$  have disjoint supports.*

As in [9, 10] a differential  $\partial : \mathcal{C} \rightarrow \mathcal{H}$  is defined by

$$\partial(a) = a \otimes \mathbf{1} \quad , \quad a \in \mathcal{C}.$$

The following properties are simple consequences of (3) and (4).

**Corollary 2.4.**

- (i)  *$\partial$  is a derivation, i.e. it is linear and*

$$\partial(fg) = (\partial f)g + f\partial g \quad , \quad f, g \in \mathcal{C}.$$

(ii)  $\partial$  is bounded, more precisely,

$$\|\partial f\|_{\mathcal{H}} = \mathcal{E}(f)^{1/2}, \quad f \in \mathcal{C}.$$

On the other hand, Eberle [13] calls a linear map  $d$  from an algebra  $\mathcal{C}$  into a direct integral  $\int_X^\oplus H_x m(dx)$  of Hilbert spaces an  $L_2$ -differential if

- (i) the span of  $\{fdg : f, g \in \mathcal{C}\}$  is dense in  $\int_X^\oplus H_x m(dx)$  and
- (ii)  $\partial(fg) = fdg + gdf$ ,  $f, g \in \mathcal{C}$ .

Recall (13) and (16). The following result is immediate.

**Corollary 2.5.**  $\partial$  is an  $L_2$ -differential on  $\mathcal{C}$ . Given  $f, g \in \mathcal{A}$ , we have  $[\partial f]_x = \Psi_x(d_x f)$  and

$$\langle \partial f, \partial g \rangle_{\mathcal{H}} = \int_X^\oplus (d_x f, d_x g)_{\mathcal{B}_x} m(dx).$$

*Remark 2.4.* (i) Similar assumptions as in [13] would allow to extend formula (13) to the entire algebra  $\mathcal{C}$ , such that each element  $f \in \mathcal{C}$  can be assigned classes  $d_x f \in \mathcal{B}_x$ ,  $x \in X$ . Then, if  $df$  denotes the measurable vector field  $x \mapsto d_x f$ ,  $f \in \mathcal{C}$ , the resulting mapping

$$d : \mathcal{C} \rightarrow \int_X^\oplus \mathcal{B}_x m(dx)$$

defines an  $L_2$ -differential. In this case also (16) extends to all of  $\mathcal{C}$  and yields an isometry  $\Psi = \int_X^\oplus \Psi_x m(dx)$  taking  $\int_X \mathcal{B}_x m(dx)$  onto  $\mathcal{H}$  such that  $\partial = \Psi \circ d$ . Note that this is closely related to the representation

$$\mathcal{H} = L_2(X, m, (\mathcal{H}_x)_{x \in X})$$

discussed in detail in Sections 3 and 4 below (see also Theorem 2.1).

- (ii) For the measurable field  $(\mathcal{H}_x)_{x \in X}$  the function  $x \mapsto d(x) = \dim \mathcal{H}_x$  from Remark 2.2 coincides with the *pointwise index* of  $(\mathcal{E}, \mathcal{F})$  as introduced by Hino in [22] (also related to the *martingale dimension of fractals*, see [21]). There a detailed analysis of pointwise and global indices is provided and applied to first order derivatives of energy finite functions on a class of fractals.

*Remark 2.5.* The above construction has utilized the energy measures (2) to generate a related algebraic structure. We would like to remind the reader of the well known fact that they also generate metric structures:

$$(19) \quad d(x, y) := \sup \left\{ f(x) - f(y) : f \in \tilde{\mathcal{C}}, \Gamma(f) \leq \mu \right\}, \quad x, y \in X,$$

where  $\tilde{\mathcal{C}}$  is a core of  $(\mathcal{E}, \mathcal{F})$  and  $\Gamma(f) \leq \mu$  stands for the requirement that  $\Gamma(f)$  is absolutely continuous with respect to  $\mu$  having density  $\frac{\Gamma(f)}{d\mu} \leq 1$   $\mu$ -a.e. provides a pseudo-metric  $d$  on  $X$ , usually referred to as *Carnot-Caratheodory distance*. If  $\tilde{\mathcal{C}}$  separates the points of  $X$ ,  $d$  is a metric in the wide sense (i.e. satisfies the axioms of a metric but may attain the value  $+\infty$ ). To our knowledge, (19) has first been considered in the context of Dirichlet forms in [7, 8, 11] and [44, 45]. Under the assumptions that  $(X, d)$  is complete and the topology induced by  $d$  on  $X$  coincides with the original one, it had been shown in [44] (together with [45]) that  $(X, d)$  is a geodesic space. In [41] the completeness assumption had been dropped. Having in mind the constructions of the present paper, it would be interesting to know whether (or for which cores  $\tilde{\mathcal{C}}$ )  $(X, d)$  is a geodesic space without any further topological assumptions.

### 3. VECTOR FIELDS, GRADIENT AND DIVERGENCE

As a Hilbert space  $\mathcal{H}$  is self-dual. We therefore regard 1-forms also as *vector fields*, exact 1-forms  $\partial f$  also *gradients* and  $\partial$  as the *gradient operator*. As  $\mathcal{C}$  is dense in  $\mathcal{F}$  which in turn is dense in  $L_2(X, \mu)$ ,  $\partial$  may be viewed as densely defined unbounded operator

$$\partial : L_2(X, \mu) \rightarrow \mathcal{H}$$

a priori equipped with the domain  $\text{dom } \partial = \mathcal{C}$ . As  $(\mathcal{E}, \mathcal{F})$  is a Dirichlet form,  $\partial$  is closable and extends uniquely to a closed linear operator  $\partial$  with domain  $\mathcal{F}$ .

In the sequel we inquire about the adjoint  $\partial^*$  of  $\partial$ . Let  $\mathcal{C}^*$  denote the dual space of  $\mathcal{C}$ , normed by

$$\|w\|_{\mathcal{C}^*} = \sup \{|w(f)| : f \in \mathcal{C}, \|f\|_{\mathcal{C}} \leq 1\}$$

and automatically a Banach space. Given  $f, g \in \mathcal{C}$ , consider the mapping

$$(20) \quad u \mapsto -\langle \partial u, g \partial f \rangle_{\mathcal{H}} = - \int_X g \, d\Gamma(u, f)$$

on  $\mathcal{C}$ . By Cauchy-Schwarz in  $\mathcal{H}$  and Corollary 2.4 (ii) we have

$$|\langle \partial u, g \partial f \rangle_{\mathcal{H}}| \leq \mathcal{E}(u)^{1/2} \|g \partial f\|_{\mathcal{H}}$$

which says that (20) defines an element  $\partial^*(g \partial f)$  of  $\mathcal{C}^*$  with norm bound

$$\|\partial^*(g \partial f)\|_{\mathcal{C}^*} \leq \|g \partial f\|_{\mathcal{H}} .$$

To

$$\partial^*(g \partial f) = - \int_X g \, d\Gamma(\cdot, f)$$

we refer as the *divergence of the vector field*  $g \partial f$ .

**Lemma 3.1.**  $\partial^*$  extends continuously to a bounded linear operator

$$\partial^* : \mathcal{H} \rightarrow \mathcal{C}^*$$

with  $\|\partial^* v\|_{\mathcal{C}^*} \leq \|v\|_{\mathcal{H}}$ ,  $v \in \mathcal{H}$ . Moreover,

$$\partial^* v(u) = -\langle \partial u, v \rangle_{\mathcal{H}}$$

for any  $u \in \mathcal{C}$  and any  $v \in \mathcal{H}$ .

$\partial^*$  will be called the *divergence operator*.

*Proof.* Let  $a_i, b_j \in \mathcal{C}$ ,  $i, j = 1, \dots, N$ . Then

$$\left\| \partial^* \left( \sum_i \sum_k a_i \otimes b_k \right) \right\|_{\mathcal{C}^*} \leq \left\| \left( \sum_i a_i \right) \otimes \left( \sum_k b_k \right) \right\|_{\mathcal{H}} .$$

Given a finite linear combination  $\sum_k g_k \partial f_k$  of simple vector fields, consider the case  $a_i = f_i$  and  $b_k = g_i$  if  $k = i$  and  $b_k = 0$  otherwise to get

$$\left\| \partial^* \left( \sum_k g_k \partial f_k \right) \right\|_{\mathcal{C}^*} \leq \left\| \sum_k g_k \partial f_k \right\|_{\mathcal{H}} .$$

Such finite linear combinations being dense in  $\mathcal{H}$ , we may extend  $\partial^*$  to the whole of  $\mathcal{H}$  with the norm bound preserved. The last assertion is an immediate consequence.  $\square$

In  $X = \mathbb{R}^n$  we have the pointwise identity

$$\operatorname{div} (g \operatorname{grad} f) = g \Delta f + \nabla f \nabla g$$

for  $f \in C^2(\mathbb{R}^n)$  and  $g \in C^1(\mathbb{R}^n)$ . Let  $(A, \operatorname{dom} A)$  denote the infinitesimal  $L_2(X, \mu)$ -generator of  $(\mathcal{E}, \mathcal{F})$ . For  $f \in \operatorname{dom} A$  and  $g, u \in \mathcal{C}$  we have

$$(21) \quad (gAf)(u) = -\mathcal{E}(gu, f),$$

and if  $f \in \mathcal{C}$ , we may use (21) as a definition of  $gAf$ : Since

$$|(gAf)(u)| \leq \mathcal{E}(gu)^{1/2} \mathcal{E}(f)^{1/2} \leq \|u\|_{\mathcal{C}} \|g\|_{\mathcal{C}} \mathcal{E}(f)^{1/2}$$

for any  $u \in \mathcal{C}$  by Cauchy-Schwarz and (1),  $gAf$  is a well defined member of  $\mathcal{C}^*$ . Similarly also the energy measure  $\Gamma(f, g)$ , seen as a linear functional

$$\Gamma(f, g)(u) := \int_X u d\Gamma(f, g)$$

on  $\mathcal{C}$ , is a member of  $\mathcal{C}^*$ , because  $\|\Gamma(f)\|_{\mathcal{C}^*} \leq \mathcal{E}(f)$  and by polarization

$$\|\Gamma(f, g)\|_{\mathcal{C}^*} \leq \frac{1}{2}(\mathcal{E}(f) + \mathcal{E}(g)).$$

**Lemma 3.2.** *For any simple vector field  $g\partial f$ ,  $f, g \in \mathcal{C}$ , we have*

$$(22) \quad \partial^*(g\partial f) = gAf + \Gamma(f, g),$$

*seen as an equality in  $\mathcal{C}^*$ . In particular,  $Af = \partial^*\partial f$  for  $f \in \mathcal{C}$ .*

*Proof.* This is now a simple consequence of the identity

$$-(gAf)(u) = \mathcal{E}(gu, f) = \int_X g d\Gamma(u, f) + \int_X u d\Gamma(f, g),$$

$u \in \mathcal{C}$ , which itself may quickly be verified using (2). □

Generally the inclusions  $\mathcal{C} \subset L_2(X, \mu) \subset \mathcal{C}^*$  are proper and seen as an operator

$$\partial^* : \mathcal{H} \rightarrow L_2(X, \mu)$$

the divergence  $\partial^*$  is unbounded. As usual  $v \in \mathcal{H}$  is said to be a member of  $\operatorname{dom} \partial^*$  if there exists some (then automatically unique)  $v^* \in L_2(X, \mu)$  such that  $\langle u, v^* \rangle_{L_2(X, \mu)} = -\langle \partial u, v \rangle_{\mathcal{H}}$  for all  $u \in \mathcal{C}$ . In this case  $\partial^*v := v^*$  and

$$(23) \quad \langle u, \partial^*v \rangle_{L_2(X, \mu)} = -\langle \partial u, v \rangle_{\mathcal{H}}, \quad u \in \mathcal{C},$$

i.e.  $-\partial^*$  is the adjoint operator of  $\partial$ . It is immediate that  $\{\partial f : f \in \operatorname{dom} A\} \subset \operatorname{dom} \partial^*$ . As  $-\partial^*$  is the adjoint of the densely defined and closable operator  $\partial$  it is densely defined, see [36].

#### 4. SOBOLEV SPACES AND COORDINATES

For a measurable section  $v = (v(x))_{x \in X}$  let

$$\|v\|_{L_p(X, m, (\mathcal{H}_x)_{x \in X})} := \left( \int_X \|v_x\|_{\mathcal{H}_x}^p m(dx) \right)^{1/p}$$

for  $1 \leq p < \infty$  and

$$\|v\|_{L_\infty(X, m, (\mathcal{H}_x)_{x \in X})} := \operatorname{ess\,sup}_{x \in X} \|v_x\|_{\mathcal{H}_x}$$

and define the spaces  $L_p(X, m, (\mathcal{H}_x)_{x \in X})$ ,  $1 \leq p \leq \infty$  as the collections of the respective equivalence classes of  $m$ -a.e. equal sections having finite norm. By a variant of the classical pointwise Riesz-Fischer argument they form Banach spaces, separable for  $1 \leq p < \infty$ . Note that  $\mathcal{H} = L_2(X, m, (\mathcal{H}_x)_{x \in X})$ .

For  $1 < p < \infty$  and  $1/p + 1/q = 1$  the Hölder inequality

$$(24) \quad \left| \int_X \langle v_x, w_x \rangle_{\mathcal{H}_x} m(dx) \right| \leq \left( \int_X \|v_x\|_{\mathcal{H}_x}^p m(dx) \right)^{1/p} \left( \int_X \|w_x\|_{\mathcal{H}_x}^q m(dx) \right)^{1/q}$$

for  $v \in L_p(X, m, (\mathcal{H}_x))$ ,  $w \in L_q(X, m, (\mathcal{H}_x))$  follows from Cauchy-Schwarz in  $\mathcal{H}$ . We will write  $\langle w, v \rangle$  for the the integral on the left hand side.

If  $f \in \mathcal{B}_b(X)$  and  $v = (v(x))_{x \in X} \in L_p(X, m, (\mathcal{H}_x))$  then the product  $fv$  is defined as the measurable section  $x \mapsto f(x)v_x$ , i.e. pointwise. Since

$$\|fv\|_{L_p(X, m, (\mathcal{H}_x)_x)} = \left( \int_X \|f(x)v_x\|_{\mathcal{H}_x}^p m(dx) \right)^{1/p} \leq \|f\|_{L_\infty(X, m)} \|v\|_{L_p(X, m, (\mathcal{H}_x)_x)}$$

the operation  $v \mapsto fv$  is linear and bounded in  $L_p(X, m, (\mathcal{H}_x))$  and is continuous with respect to the pointwise convergence of uniformly bounded sequences, i.e. if  $\sup_n \|f_n\|_{L_\infty(X, m)} < \infty$  and  $\lim_n f_n = f$  pointwise  $m$ -a.e. on  $X$ , then  $\lim_n f_n v = fv$  in  $L_p(X, m, (\mathcal{H}_x))$  for all  $v \in L_p(X, m, (\mathcal{H}_x))$ . For  $p = 2$  this multiplication coincides with (5).

In the sequel we will assume that for any  $1 < p < \infty$  there is a space  $\mathcal{C}_p \subset \mathcal{C} \cap L_p(X, m)$  such that

(COREI)  $\mathcal{C}_p$  is dense in  $L_p(X, m)$ ,

(COREII)  $\mathcal{C}_p \otimes \mathcal{C}_p$  is dense in  $L_p(X, m, (\mathcal{H}_x)_x)$  and

(COREIII) for all  $f \in \mathcal{C}_p$ , the energy measure  $\Gamma(f)$  is absolutely continuous with respect to  $m$  with density

$$\Gamma(f) = \frac{d\Gamma(f)}{dm} \in L_{p/2}(X, m).$$

Let  $\partial_p$  denote the restriction of  $\partial$  to  $\mathcal{C}_p$ . By (COREIII)  $\partial_p$  maps  $\mathcal{C}_p$  into  $L_p(X, m, (\mathcal{H}_x)_x)$ . (COREII) implies closability:

**Lemma 4.1.** *( $\partial_p, \mathcal{C}_p$ ) is closable for any  $1 < p < \infty$ .*

*Proof.* Let  $(f_n) \subset \mathcal{C}_p$  be a sequence of functions converging to zero in  $L_p(X, m)$  and such that  $(\partial_p f_n)$  is Cauchy in  $L_p(X, m, (\mathcal{H}_x))$ . As the latter space is complete, a unique limit  $v := \lim_n \partial_p f_n \in L_p(X, m, (\mathcal{H}_x))$  exists. For an arbitrary member  $f \otimes g$  of  $\mathcal{C}_q \otimes \mathcal{C}_q$  with  $1/p + 1/q = 1$  we have

$$\langle f \otimes g, v \rangle = \lim_n \langle f \otimes g, \partial_p f_n \rangle = \lim_n \langle f \otimes g, \partial f_n \rangle_{\mathcal{H}} = - \lim_n \partial^*(g \partial f)(f_n) = 0$$

by Lemma 3.1 and because  $\partial^*(f \otimes g) \in \mathcal{C}^*$  according to Lemma 3.2. By (COREII) therefore  $\lim_n \partial_p f_n = 0$  in  $L_p(X, m, (\mathcal{H}_x))$ .  $\square$

Denote the smallest closed extension of  $(\partial_p, \mathcal{C}_p)$  by  $(\partial_p, \text{dom } \partial_p)$ , by (COREI) it is a densely defined closed linear operator from  $L_p(X, m)$  into  $L_p(X, m, (\mathcal{H}_x))$ . Note that for any simple vector field  $g\partial f$  with  $f, g \in \mathcal{C}_p$  we then have

$$(25) \quad \|g\partial f\|_{L_p(X, m, (\mathcal{H}_x)_x)} = \left( \int_X |g(x)|^p \Gamma_x(f)^{p/2} m(dx) \right)^{1/p}.$$

Given  $1 < p < \infty$  write  $H_0^{1,p}(X, m)$  for  $\text{dom } \partial_p$ , equipped with the norm

$$\|u\|_{1,p} := \left( \int_X (|u(x)|^p + \|\partial_x u\|_{\mathcal{H}_x}^p) m(dx) \right)^{1/p}, u \in H_0^{1,p}(X, m).$$

As  $\|\cdot\|_{1,p}$  is equivalent to the graph norm of  $\partial_p$ ,  $H_0^{1,p}(X, m)$  is a closed subspace of  $L_p(X, m)$ , clearly Banach, and continuously embedded in  $L_p(X, m)$ .

*Remark 4.1.*

- (i) (COREI)-(COREIII) in particular imply that  $\mathcal{C}_2$  is dense,  $(\partial_2, \mathcal{C}_2)$  is closable and therefore  $(\mathcal{E}, H_0^{1,2}(X, m))$  is a Dirichlet form on  $L_2(X, m)$ .
- (ii) In general  $L_2(X, \mu) \neq L_2(X, m)$  and  $H_0^{1,2}(X, m) \neq \mathcal{F}$ . However, if  $m$  and  $\mu$  are equivalent with bounded densities, then these spaces agree and  $\|u\|_{1,2}$  is equivalent to  $\mathcal{E}_1(u)^{1/2}$ .

Let  $1 < p < \infty$ . The divergence operator  $\partial^*$  may be seen as an unbounded operator  $\partial_q^*$  from  $L_q(X, m, (\mathcal{H}_x))$  into  $L_q(X, m)$ ,  $1/p + 1/q = 1$ , and similarly as in (23) we obtain an integration by parts formula by saying that an element  $v \in L_q(X, m, (\mathcal{H}_x)_{x \in X})$  is in  $\text{dom } \partial_q^*$  if there is some  $v^* \in L_q(X, m)$  such that  $\langle u, v^* \rangle = -\langle \partial u, v \rangle$  for all  $u \in \mathcal{C}_p$ . We write  $\partial_q^* v := v^*$  and

$$\langle u, \partial_q^* v \rangle = -\langle \partial u, v \rangle, u \in \mathcal{C}_p.$$

By duality  $\text{dom } \partial_q^*$  is then weakly dense in  $L_q(X, m)$ , cf. [36].

*Examples 4.1.* If  $X \subset \mathbb{R}^n$  is a sufficiently nice domain,  $m(dx) = \mu(dx) = dx$  is the  $n$ -dimensional Lebesgue measure on  $X$  and

$$\mathcal{E}(f, g) = \int_X \nabla f \nabla g dx$$

is the Dirichlet form of the standard Laplacian, all fibers  $\mathcal{H}_x = \mathbb{R}^n$  are constant, a canonical choice for  $\mathcal{C}_p$  is  $C_0^\infty(X)$  and  $\Gamma(f) = |\nabla f|^2$ . Similarly for more general second order elliptic differential operators.

A method to prove assumptions (COREI)-(COREIII) in some non-classical contexts is the following. Let us say that  $\mathcal{E}$  *admits (continuous) coordinates* if there exist a set  $I \neq \emptyset$  and a collection of functions  $\{\varphi_i\}_{i \in I} \subset \mathcal{C}$  such that

- (COI) for all  $i, j \in I$ ,  $\Gamma(\varphi_i, \varphi_j) \in L_1(X, m) \cap L_\infty(X, m)$ ,
- (COII)  $\{\varphi_i\}_{i \in I}$  separates the points of  $X$  and vanishes nowhere, i.e. for any two distinct points  $x, y \in X$  there exist some  $i, j \in I$  such that  $\varphi_i(x) \neq \varphi_i(y)$  and  $\varphi_j(x) \neq 0$ ,

(COIII) The set  $\mathcal{FC}_b^1(X, \{\varphi_i\})$  of all cylinder functions of form

$$f = F(\varphi_{i_1}, \dots, \varphi_{i_m}), \quad i_1, \dots, i_m \in I$$

with suitable  $k \in \mathbb{N}$  and  $F \in C_b^1(\mathbb{R}^k)$  is  $\mathcal{E}$ -dense in  $\mathcal{C}$ .

A function  $f \in \mathcal{C}$  is said to be in  $\text{dom } A_m$  if there is a function  $A_m f \in L_2(X, m)$  such that

$$\mathcal{E}(f, g) = -\langle g, A_m f \rangle_{L_2(X, m)}$$

for all  $g \in \mathcal{C}$ .

**Lemma 4.2.** Suppose  $\{\varphi_i\}_{i \in I} \subset \text{dom } A_m \cap L_1(X, m)$  such that for all  $i, j \in I$ ,  $A_m \varphi_i \in L_1(X, m) \cap L_\infty(X, m)$  as well as  $\varphi_i \varphi_j \in \text{dom } A_m$  and  $A_m(\varphi_i \varphi_j) \in L_1(X, m) \cap L_\infty(X, m)$ . Then condition (COI) follows.

*Proof.* In this case  $2\mathcal{E}(\varphi_i h, \varphi_i) - \mathcal{E}(h, \varphi_i^2) = \int_X h(2\varphi_i A_m \varphi_i - A_m \varphi_i^2) dm$  for all  $h \in \mathcal{C}$ , cf. [6], and consequently for all  $i, j \in I$ ,  $\Gamma(\varphi_i, \varphi_j) = -A_m(\varphi_i \varphi_j) + \varphi_i A_m \varphi_j + \varphi_j A_m \varphi_i$  by polarization. Then, according to our hypotheses,  $\Gamma(\varphi_i, \varphi_j) \in L_1(X, m) \cap L_\infty(X, m)$ .  $\square$

*Examples 4.2.* Let  $(\mathcal{E}, \mathcal{F})$  be a resistance form, [26], on a finitely ramified fractal  $X$ , which is defined as a specific type of cell-structured compact topological space, see [47] for the precise definition. Let  $\varphi_1, \dots, \varphi_k$  be a complete, up to constants, energy orthonormal set of harmonic functions and define a finite reference measure by

$$(26) \quad m := \sum_{j=1}^k \Gamma(\varphi_j)$$

(*Kusuoka measure*), where  $\Gamma(\varphi_j)$  are the energy measures of the functions  $\varphi_j$ . A function  $f$  is said to be in the domain of the associated Laplacian  $\Delta_m$  if there exists a function  $\Delta_m f \in L_2(X, m)$  with

$$\mathcal{E}(f, g) = - \int_X g \Delta_m f dm$$

for any  $g \in \mathcal{F}$  vanishing on the boundary (in the sense on finitely ramified fractals), which yields a weak definition for the Laplacian  $\Delta_m$  as the mapping  $f \mapsto \Delta_m f$ . Consider the coordinate map  $\psi : X \rightarrow \mathbb{R}^m$ ,  $\psi(x) = (\varphi_1(x), \dots, \varphi_k(x))$ , cf. [47]. We assume that  $\psi : X \rightarrow \psi(X)$  is a homeomorphism. This implies that all  $\varphi_j$  are continuous on  $X$ , cf. [47, Proposition 5.3]. It also implies that they separate points. By construction they vanish nowhere. Clearly  $\varphi_1, \dots, \varphi_k \in \text{dom } \Delta_m \cap L_\infty(X, m)$  and  $\Delta_m \varphi_j = 0$  for all  $j$ . By [47, Theorem 8 respectively Corollary 6.1],  $\varphi_i \varphi_j \in \text{dom } \Delta_m$  and  $\Delta_m(\varphi_i \varphi_j) \in L_\infty(X, m)$ . Consequently  $\mathcal{E}$  admits coordinates by Lemma 4.2.

Given cylinder functions  $f = F(\varphi_{i_1}, \dots, \varphi_{i_m})$  and  $g = G(\varphi_{j_1}, \dots, \varphi_{j_n})$  with  $F \in C_b^1(\mathbb{R}^m)$ ,  $G \in C_b^1(\mathbb{R}^n)$ ,

$$\Gamma(f, g) = \sum_{k=1}^m \sum_{l=1}^n \frac{\partial F}{\partial x_k}(\varphi_{i_1}, \dots, \varphi_{i_m}) \frac{\partial G}{\partial x_l}(\varphi_{j_1}, \dots, \varphi_{j_n}) \Gamma(\varphi_k, \varphi_l)$$

by the chain rule, cf. [16], Theorem 3.2.2. In particular,  $\Gamma(f, g)$  is a member of  $L_\infty(X, m)$  and has compact support. Obviously  $\mathcal{FC}_b^1(X, \{\varphi_i\})$  is a subalgebra of  $C_\infty(X)$ , and by the Stone-Weierstrass theorem together with the denseness of  $C_0(X)$  in  $L_p(X, m)$  we obtain the following.



**Corollary 4.1.** Assume that  $\mathcal{E}$  admits coordinates  $\{\varphi_i\}_{i \in I}$ . Then for any  $1 \leq p < \infty$  assumptions (COREI) and (COREIII) are satisfied with  $\mathcal{C}_p = \mathcal{FC}_b^1(X, \{\varphi_i\})$ .

Another consequence concerns the spaces  $L_p(X, m, (\mathcal{H}_x)_x)$ .

**Lemma 4.3.** Assume that  $\mathcal{E}$  admits coordinates  $\{\varphi_i\}_{i \in I}$ . Then for any  $1 < p < \infty$ ,

$$\mathcal{S} := \text{span} \{g \partial f : f, g \in \mathcal{FC}_b^1(X, \{\varphi_i\})\}$$

is a dense subspace of  $L_p(X, m, (\mathcal{H}_x)_x)$ . In particular, (CORE II) holds with  $\mathcal{C}_p = \mathcal{FC}_b^1(X, \{\varphi_i\})$ .

*Proof.* By (25) it is easily seen that for any  $1 < p < \infty$ ,  $\mathcal{S}$  is a subspace of  $L_p(X, m, (\mathcal{H}_x)_x)$ . Next observe that

$$\mathcal{S}_0 := \text{span} \{g \partial f : f \in \mathcal{FC}_b^1(X, \{\varphi_i\}), g \in \mathcal{B}_b(X)\}$$

is dense in  $\mathcal{H} = L_2(X, m, (\mathcal{H}_x)_x)$  because, by the definition of  $\mathcal{H}$ , it suffices to approximate finite linear combinations  $\sum_i a_i \otimes b_i \in \mathcal{C} \otimes \mathcal{B}_b(X)$ . For fixed  $i$ , let  $(a_i^{(m)})_m \subset \mathcal{FC}_b^1(X, \{\varphi_i\})$  be a sequence approximating  $a_i$  in  $\mathcal{E}$ . Then

$$\left\| \sum_i a_i \otimes b_i - \sum_i a_i^{(m)} \otimes b_i \right\|_{\mathcal{H}}^2 = \sum_i \sum_j \int_X b_i b_j d\Gamma(a_i - a_i^{(m)}),$$

which is bounded by  $2 \max_i \sup_X |b_i|^2 \sum_i \mathcal{E}(a_i - a_i^{(m)})$  and therefore converges to zero as  $m$  goes to infinity.

$\mathcal{S}_0$  is dense in  $L_p(X, m, (\mathcal{H}_x)_x)$ : Assume it were not, then by Hahn-Banach we could find some  $\eta \in L_q(X, m, (\mathcal{H}_x)_x)$ ,  $1/p + 1/q = 1$ , such that  $\|\eta\|_{L_q(X, m, (\mathcal{H}_x)_x)} = 1$  and

$$(27) \quad \langle \omega, \eta \rangle = 0 \quad \text{for all } \omega \in \mathcal{S}_0.$$

For any  $N \in \mathbb{N}$  let  $K_N \subset X$  be compact such that  $m(X \setminus K_N) < 1/N$  and set

$$S_N := \{x \in X : \|\eta_x\|_{\mathcal{H}_x} < N\} \cap K_N.$$

Then

$$\lim_N \|\eta \mathbf{1}_{S_N} - \eta\|_{L_q(X, m, (\mathcal{H}_x)_x)}^q = \lim_N \int_X |\mathbf{1}_{S_N}(x) - \mathbf{1}(x)|^q \|\eta_x\|_{\mathcal{H}_x}^q m(dx) = 0$$

by dominated convergence and accordingly for fixed  $\varepsilon > 0$  there exists some  $N_\varepsilon > 0$  such that for any  $N \geq N_\varepsilon$ ,  $\|\eta \mathbf{1}_{S_N} - \eta\|_{L_q(X, m, (\mathcal{H}_x)_x)} < \varepsilon$ . Note also that  $\eta \mathbf{1}_{S_N} \in \mathcal{H}$  for all  $N \in \mathbb{N}$  since

$$\int_X \|\eta_x \mathbf{1}_{S_N}(x)\|_{\mathcal{H}_x}^2 m(dx) < N^2 m(K_N) < \infty.$$

As  $\|\eta \mathbf{1}_{S_N}\|_{L_q(X, m, (\mathcal{H}_x)_x)} > 1 - \varepsilon$  necessarily also

$$\delta_N := \|\eta \mathbf{1}_{S_N}\|_{\mathcal{H}} > 0.$$

Now let  $(\omega_n)_n \subset \mathcal{S}_0$ ,  $\omega_n = \sum_i a_i^{(n)} \otimes b_i^{(n)}$  be a sequence that approximates  $\eta \mathbf{1}_{S_N}$  in  $\mathcal{H}$ . Let  $0 < \gamma < \delta_N$  and  $n \in \mathbb{N}$  be so large that

$$\|\eta \mathbf{1}_{S_N} - \omega_n\|_{\mathcal{H}} \leq \gamma.$$

Since  $|\langle \mathbf{1}_{S_N} \eta, \mathbf{1}_{S_N} \eta - \omega_n \rangle| \leq \gamma \delta_N$  we obtain

$$(28) \quad |\langle \mathbf{1}_{S_N} \omega, \eta \rangle| = |\langle \omega, \mathbf{1}_{S_N} \eta \rangle| > \delta_N (\delta_N - \gamma) > 0.$$

On the other hand

$$\mathbf{1}_{S_N} \omega_n = \sum_i \mathbf{1}_{S_N} (a_i^{(n)} \otimes b_i^{(n)}) = \sum_i a_i^{(n)} \otimes (\mathbf{1}_{S_N} b_i^{(n)})$$

itself is an element of  $\mathcal{S}_0$ . Therefore (28) contradicts (27).

$\mathcal{S}$  in turn is  $L_p(X, m, (\mathcal{H}_x)_x)$ -dense in  $\mathcal{S}_0$ : Let  $\sum_i a_i \otimes b_i \in \mathcal{FC}_b^1(X, \{\varphi_i\}) \otimes \mathcal{B}_b(X)$ . Any  $b_i$  can be approximated uniformly by a sequence  $(b_i^{(m)})_m$  from  $C_\infty(X)$ , and by Stone-Weierstrass this sequence may be taken from  $\mathcal{FC}_b^1(X, \{\varphi_i\})$ . Then by (25),

$$\left\| \sum_i a_i \otimes b_i - \sum_i a_i \otimes b_i^{(m)} \right\|_{L_p(X, m, (\mathcal{H}_x)_x)} \leq \sum_i \left( \int_X |b_i(x) - b_i^{(m)}(x)|^p \Gamma_x(a_i)^{p/2} m(dx) \right)^{1/p},$$

clearly converging to zero.

□

*Remark 4.2.* If  $\mathcal{E}$  admits coordinates then  $(\mathcal{E}, \mathcal{FC}_b^1(X, \{\varphi_i\}))$  is closable and the choice  $\mathcal{C}_2 := \mathcal{FC}_b^1(X, \{\varphi_i\})$  yields a regular Dirichlet form  $(\mathcal{E}, H_0^{1,2}(X, m))$  on  $L_2(X, m)$ , cf. Remark 4.1.

We conclude this section with a brief look at the related  $p$ -energy. For fixed  $1 < p < \infty$  the mapping

$$f \mapsto \mathcal{E}_p(f) := \int_X \Gamma(f)^{p/2} dm, \quad f \in H_0^{1,p}(X, m),$$

is usually referred to as the  $p$ -energy functional. One may define a functional of two arguments by

$$(29) \quad \mathcal{E}_p(f, g) := \int_X \Gamma(f)^{p/2-1} \Gamma(f, g) dm, \quad f, g \in H_0^{1,p}(X, m).$$

Note that  $\mathcal{E}_p(f, f) = \mathcal{E}_p(f)$  and that by Hölder's inequality,  $|\mathcal{E}_p(f, g)| \leq \mathcal{E}_p(f)^{(p-1)/p} \mathcal{E}_p(g)^{1/p}$ . For sufficiently nice functions  $\varphi, \psi$  (for instance for  $\varphi, \psi \in \text{dom } A_m$  satisfying the hypotheses of Lemma 4.2 in place of  $\varphi_i$  and  $\varphi_j$ ) we observe

$$\mathcal{E}_p(\varphi, \psi) := \frac{1}{p} \frac{d}{dt} \mathcal{E}_p(\varphi + t\psi)|_{t=0}.$$

A generalized  $p$ -Laplacian may be defined in the weak sense by associating to  $f \in H_0^{1,p}(X, m)$  the element  $\Delta_p f$  of the dual space  $(H_0^{1,p}(X, m))^*$  given by

$$(\Delta_p f)(g) := -\mathcal{E}_p(f, g) = - \int_X \|\partial_x f\|_{\mathcal{H}_x}^{p-2} \langle \partial_x f, \partial_x g \rangle_{\mathcal{H}_x} m(dx) = - \langle \|\partial_x f\|_{\mathcal{H}_x}^{p-2} \partial f, \partial g \rangle_{\mathcal{H}},$$

$g \in H_0^{1,p}(X, m)$ . Integrating by parts we obtain

$$\Delta_p f = \partial_p^* (\|\partial_x f\|_{\mathcal{H}_x}^{p-2} \partial f).$$

If  $A = \Delta$  is the classical Laplacian on  $\mathbb{R}^n$  and  $m(dx) = dx$  the  $n$ -dimensional Lebesgue measure, then  $\Delta_p$  is the usual  $p$ -Laplacian.

*Remark 4.3.* Another definition for a  $p$ -energy on Sierpinski gasket type fractals had been proposed in [18], a related  $p$ -Laplacian had been investigated in [43]. To attempt a comparison, we first rewrite our previous definition. For simplicity, let  $X = K$  be the Sierpinski gasket  $K = \bigcup_{i=1}^3 F_i K$ , where  $\{F_1, F_2, F_3\}$  is the iterated function system of contractive similarities  $F_i$  with common contraction ratio  $\frac{1}{2}$  and fixed points  $q_1, q_2, q_3$  which are the vertices

of an equilateral triangle. Let  $V_0 = \{q_1, q_2, q_3\}$  and  $V_m = \bigcup_{i=1}^3 F_i V_{m-1}$ ,  $m \geq 1$ . Write  $F_w$  for  $F_{w_1} \cdots F_{w_n}$  and a word  $w = (w_1, \dots, w_n)$  of length  $|w| = n$  over the alphabet  $\{1, 2, 3\}$ . By Kusuoka's construction, [27, 29, 47], we have

$$\mathcal{E}(f) = \int_K \Gamma(f) dm = \int_K \langle \nabla F(\varphi_1(x), \varphi_2(x)), Z_x \nabla F(\varphi_1(x), \varphi_2(x)) \rangle_{\mathbb{R}^2} dm$$

for functions  $f = F(\varphi_1, \varphi_2)$ , where  $\{\varphi_1, \varphi_2\}$  is an energy orthonormal basis of harmonic functions and  $F \in C_b^1(\mathbb{R}^2)$ . The measure  $m = \Gamma(\varphi_1) + \Gamma(\varphi_2)$  is the Kusuoka measure as in (26).  $\nabla F$  is the usual gradient of  $F$  in  $\mathbb{R}^2$  and  $Z = (Z_x)_{x \in K}$  is a measurable  $(2 \times 2)$ -matrix valued function on  $K$  such that  $\text{rank } Z = 1$  and  $\text{Tr } Z = 1$   $m$ -a.e. It arises as the  $m$ -a.e. limit of a bounded  $(2 \times 2)$ -matrix valued  $m$ -martingale  $(Z_n)_{n \in \mathbb{N}}$ , cf. [29, 47]. Accordingly for fixed  $f$  as above,  $(\langle \nabla F(\varphi_1, \varphi_2), Z_n \nabla F(\varphi_1, \varphi_2) \rangle_{\mathbb{R}^2})_{n \in \mathbb{N}}$  is a bounded  $m$ -martingale with  $m$ -a.s. limit  $\langle \nabla F(\varphi_1, \varphi_2), Z \nabla F(\varphi_1, \varphi_2) \rangle_{\mathbb{R}^2}$ . We have

$$\mathcal{E}(f) = \lim_{n \rightarrow \infty} \int_K \langle \nabla F(\varphi_1, \varphi_2), Z_n \nabla F(\varphi_1, \varphi_2) \rangle_{\mathbb{R}^2} dm$$

and

$$\langle \nabla F(\varphi_1, \varphi_2), Z_n \nabla F(\varphi_1, \varphi_2) \rangle_{\mathbb{R}^2} = \sum_{|w|=n} \mathbf{1}_{F_w K} \frac{r^{-n} \sum_{i=1}^3 (f(F_w q_i) - f(F_w q_{i+1}))^2}{m(F_w K)}$$

(where  $q_4 := q_1$ ) for  $f = F(\varphi_1, \varphi_2)$ ,  $F \in C^1(\mathbb{R}^2)$ . Here  $r = \frac{3}{5}$ . By bounded convergence also

$$\begin{aligned} \mathcal{E}_p(f) &= \int_K \Gamma(f)^{p/2} dm \\ &= \lim_{n \rightarrow \infty} \int_K \langle \nabla F(\varphi_1, \varphi_2), Z_n \nabla F(\varphi_1, \varphi_2) \rangle_{\mathbb{R}^2}^{p/2} dm \\ &= \lim_{n \rightarrow \infty} \sum_{|w|=n} \left( \frac{r^{-n} \sum_{i=1}^3 (f(F_w q_i) - f(F_w q_{i+1}))^2}{m(F_w K)} \right)^{p/2} m(F_w K), \end{aligned}$$

and each member of this sequence is comparable to

$$r^{-np/2} \sum_{|w|=n} \frac{\sum_{i=1}^3 |f(F_w q_i) - f(F_w q_{i+1})|^p}{m(F_w K)^{p/2}} m(F_w K).$$

The  $p$ -energy in [18] had been constructed by solving an abstract renormalization problem whose solution allows to define the  $p$ -energy as the limit of an increasing sequence of approximative  $p$ -energies on the sets  $V_m$ . Each member of this sequence is comparable to

$$r_p^{-n} \sum_{|w|=n} \frac{\sum_{i=1}^3 |f(F_w q_i) - f(F_w q_{i+1})|^p}{m(F_w K)} m(F_w K),$$

where  $r_p$  is a scaling factor that is part of the solution of the renormalization problem. The construction is not very explicit and therefore it seems difficult to read off specific properties. However, since it is known that along different infinite words  $w = w_1 w_2 w_3 \dots$  the quantity  $m(F_{w|n} K)$ , where  $w|_n = w_1 \dots w_n$ , has different growth behaviour, cf. [5], one cannot expect the ratios

$$\frac{r^{-np/2} m(F_{w|n} K)^{1-p/2}}{r_p^{-n}}$$

to have nontrivial and finite limits simultaneously for all infinite words  $w$  as  $n$  goes to infinity. Accordingly the domains of the two  $p$ -energies will most likely be disjoint.

## 5. APPLICATIONS TO PDE AND SPDE

In this section the discussed setup is used to solve PDE and SPDE by fixed point and monotonicity arguments. We focus on equations involving terms  $u \mapsto \operatorname{div} a(\operatorname{grad} u)$  and  $u \mapsto b(\nabla u)$ , where  $a$  and  $b$  are possibly nonlinear transformations. In our context these expressions rewrite  $u \mapsto \partial^*(a(\partial u))$  and  $u \mapsto b(\partial u)$ , respectively.

*Quasilinear elliptic PDE in divergence form.* Consider the quasilinear PDE

$$(30) \quad \partial^* a(\partial u) = f.$$

One possible setup to study (30) is given by the Hilbert space  $L_2(X, \mu)$ . In this case  $f$  should be an element of  $L_2(X, \mu)$  and the gradient  $\partial$  and divergence  $\partial^*$  should be interpreted as in Section 3.

An alternative setup is to view (30) in  $L_2(X, m)$ . This is possible whenever  $\mathcal{E}$  admits coordinates, recall Remark 4.2. Then  $\partial = \partial_2$  and  $\partial^* = \partial_2^*$  are to be interpreted as in Section 4 and  $f$  is to be taken from  $L_2(X, m)$ . For convenience we write  $\nu := \mu$  and  $H_0^{1,2}(X, \nu) := \mathcal{F}$  in the first case and  $\nu := m$  and  $H_0^{1,2}(X, \nu) := H_0^{1,2}(X, m)$  in the second.

$\operatorname{Im} \partial$  denotes the image of  $H_0^{1,2}(X, \nu)$  under  $\partial$ , clearly a closed subspace of  $\mathcal{H}$ .

Assume that  $a : \mathcal{H} \rightarrow \mathcal{H}$  satisfies the following monotonicity, growth and coercivity conditions:

$$(31) \quad \langle a(v) - a(w), v - w \rangle_{\mathcal{H}} \geq 0 \quad \text{for all } v, w \in \operatorname{Im} \partial,$$

$$(32) \quad \|a(v)\|_{\mathcal{H}} \leq c_0(1 + \|v\|_{\mathcal{H}}) \quad \text{for all } v \in \operatorname{Im} \partial$$

with some constant  $c_0 > 0$ ,

$$(33) \quad \langle a(v), v \rangle_{\mathcal{H}} \geq c_1 \|v\|_{\mathcal{H}}^2 - c_2 \quad \text{for all } v \in \operatorname{Im} \partial$$

with constants  $c_1 > 0$ ,  $c_2 \geq 0$ . Finally, suppose the validity of a *Poincaré inequality*,

$$(34) \quad \|f\|_{L_2(X, \nu)}^2 \leq c_P \mathcal{E}(f)$$

with some constant  $c_P > 0$  for all  $f \in L_2(X, \nu)$  with  $\int_X f d\nu = 0$ . A function  $u \in H_0^{1,2}(X, \nu)$  is a *weak solution to (30)* if

$$\langle a(\partial u), \partial v \rangle_{\mathcal{H}} = - \langle f, v \rangle_{L_2(X, \nu)} \quad \text{for all } v \in H_0^{1,2}(X, \nu).$$

The classical Brouwer-Minty monotonicity arguments based on Schauder's fixed point theorem, cf. [14, Section 9.1], now yield the following:

**Theorem 5.1.** *Assume  $a$  satisfies (31), (32) and (33) and suppose (34) holds. Then (30) has a weak solution. Moreover, if  $a$  is strictly monotone,*

$$(35) \quad \langle a(v) - a(w), v - w \rangle_{\mathcal{H}} \geq c_3 \|v - w\|_{\mathcal{H}}^2 \quad \text{for all } v, w \in \mathcal{H}$$

*with some constant  $c_3 > 0$ , then (30) has a unique weak solution.*

*Remark 5.1.* If  $a$  is a decomposable (non-linear) operator, that is if  $a = (a_x)_{x \in X}$  with  $a_x : \mathcal{H}_x \rightarrow \mathcal{H}_x$ ,  $x \in X$  and  $m - \text{ess sup}_{x \in X} \|a_x\|_{\mathcal{H}_x \rightarrow \mathcal{H}_x} < \infty$ , then to have (35) it is sufficient to have

$$\langle a_x(v(x)) - a_x(w(x)) \rangle_{\mathcal{H}_x} \geq c_4 \|v(x) - w(x)\|_{\mathcal{H}_x}$$

with a constant  $c_4 > 0$  for all  $v, w \in \mathcal{H}$  and  $m$ -a.e.  $x \in X$ . Likewise for conditions (31)-(33).

*Quasilinear elliptic PDE in non-divergence form.* Consider the PDE

$$(36) \quad -Au + b(\partial u) + \varrho u = 0,$$

where  $\varrho > 0$  and  $b$  is a generally non-linear function-valued mapping on  $\mathcal{H}$ . Also (36) can be viewed in the two different setups mentioned above, and we follow the same notational convention. In particular, we assume that  $b : \mathcal{H} \rightarrow L_2(X, \nu)$  such that

$$(37) \quad \|b(v)\|_{L_2(X, \nu)} \leq c_5(1 + \|v\|_{\mathcal{H}}), \quad v \in \text{Im } \partial,$$

with some  $c_5 > 0$ . A function  $u \in H_0^{1,2}(X, \nu)$  is called a weak solution to (36) if

$$\mathcal{E}(u, v) + \langle b(\partial u), \partial v \rangle_{\mathcal{H}} + \varrho \langle u, v \rangle_{L_2(X, \nu)} = 0$$

for all  $v \in H_0^{1,2}(X, \nu)$ . Along the lines of [14, Section 9.2.2, Example 2], we obtain the following.

**Theorem 5.2.** *Assume that the embedding  $H_0^{1,2}(X, \nu) \subset L_2(X, \nu)$  is compact and that (37) holds. Then for any sufficiently large  $\varrho > 0$  there exists a weak solution to (36).*

For convenience we briefly comment on the proof.

*Proof.* Given  $u \in H_0^{1,2}(X, \nu)$ , note that  $-b(\partial u) \in L_2(X, \nu)$  and denote by  $w$  the unique weak solution to the linear problem  $-Aw + \varrho w = -b(\partial u)$ , i.e. the unique function  $w \in H_0^{1,2}(X, \nu)$  such that

$$(38) \quad \mathcal{E}(w, v) + \varrho \langle w, v \rangle_{L_2(X, \nu)} = -\langle b(\partial u), v \rangle_{L_2(X, \nu)}$$

for all  $v \in H_0^{1,2}(X, \nu)$ . From (37) we obtain  $\|Aw\|_{L_2(X, \nu)} \leq c(1 + \|u\|_{1,2})$ , where  $\|\cdot\|_{1,2}$  denotes the norm in  $H_0^{1,2}(X, \nu)$ . By the compact embedding, the mapping  $u \mapsto \Phi(u) := w$  is easily seen to be continuous and compact from  $H_0^{1,2}(X, \nu)$  into itself. See [14, Section 9.2.2, Theorem 5]. The set

$$\{u \in H_0^{1,2}(X, \nu) : u = \lambda \Phi(u) \text{ for some } 0 < \lambda \leq 1\}$$

is bounded in  $H_0^{1,2}(X, \nu)$ : For a member of this set, (38) implies

$$\begin{aligned} \mathcal{E}(u) + \varrho \|u\|_{L_2(X, \nu)}^2 &= -\lambda \langle b(\partial u), u \rangle_{L_2(X, \nu)} \\ &\leq \|b(\partial u)\|_{L_2(X, \nu)} \|u\|_{L_2(X, \nu)} \\ &\leq c_5 \varepsilon (1 + \|\partial u\|_{\mathcal{H}}) \varepsilon^{-1} \|u\|_{L_2(X, \nu)} \\ &\leq c_5 (\varepsilon + \varepsilon \mathcal{E}(u)^{1/2} + \varepsilon^{-1} \|u\|_{L_2(X, \nu)})^2 \\ &\leq c(1 + \varepsilon^2 \mathcal{E}(u) + \varepsilon^{-2} \|u\|_{L_2(X, \nu)}^2) \end{aligned}$$

for any  $\varepsilon > 0$  and with a constant  $c > 0$  independent of  $\varepsilon$ ,  $\lambda$  and  $u$ . Now choose  $\varepsilon > 0$  sufficiently small and  $\varrho > 0$  sufficiently large to obtain

$$\|u\|_{1,2} \leq 2c.$$

Altogether this allows the application of Schaefer's fixed point theorem, cf. [14, Section 9.2.2, Theorem 4], to obtain the existence of a fixed point  $u = \Phi(u)$  in  $H_0^{1,2}(X, \nu)$ .  $\square$

*SPDE in variational form.* Another class of nicely tractable examples is provided by SPDE in variational form as studied by [28] and [33], see for instance the exposition in [35]. Here we assume throughout that  $\mathcal{E}$  admits coordinates. For simplicity consider Itô SPDE with additive Brownian noise of type

$$(39) \quad du(t) = \partial^* a(\partial u(t)) dt + \sqrt{Q} dW(t)$$

on  $(0, T) \times X$  with some initial condition  $u(0) = u_0$ . Here  $(W(t))_{t \geq 0}$  is a cylindrical Wiener process on  $L_2(X, m)$  of form

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k$$

where  $\{\beta_k\}$  is a sequence of mutually independent one-dimensional standard Brownian motions on a filtered complete probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ , and  $\{e_k\}$  is an orthonormal basis in  $L_2(X, m)$ . In the sequel let  $2 \leq p < \infty$  and assume  $m(X) < \infty$  if  $p > 2$ . Then  $L_p(X, m) \subset L_2(X, m)$ , and we have

$$H_0^{1,p}(X, m) \subset L_2(X, m) \subset (H_0^{1,p}(X, m))^*,$$

where as before  $(H_0^{1,p}(X, m))^*$  denotes the dual space of  $H_0^{1,p}(X, m)$ , and the embeddings are continuous. We write

$$\langle u, v \rangle := v(u) \quad \text{for } u \in (H_0^{1,p}(X, m))^* \text{ and } v \in H_0^{1,p}(X, m).$$

Generalizing the growth condition (32) we may require  $a$  to be a bounded operator

$$a : L_p(X, m, (\mathcal{H}_x)_x) \rightarrow L_q(X, m, (\mathcal{H}_x)_x)$$

with  $1/p + 1/q = 1$  and such that

$$(40) \quad \|a(v)\|_{L_q(X, m, (\mathcal{H}_x)_x)} \leq c_0(1 + \|v\|_{L_p(X, m, (\mathcal{H}_x)_x)}^{p-1})$$

for all  $v \in L_p(X, m, (\mathcal{H}_x)_x)$ .

*Remark 5.2.* (40) is obviously valid with  $p = 2$  if  $a = (a_x)_x$  with bounded operators  $a_x : \mathcal{H}_x \rightarrow \mathcal{H}_x$  such that  $\text{m-ess sup}_{x \in X} \|a_x\|_{\mathcal{H}_x \rightarrow \mathcal{H}_x} < \infty$ .

The following is a simple consequence of the Hölder inequality (24):

**Lemma 5.1.** *If  $a$  satisfies (40) then  $\partial^* a(\partial \cdot)$  defines a bounded operator from  $H_0^{1,p}(X, m)$  into  $(H_0^{1,p}(X, m))^*$  with*

$$\|\partial^* a(\partial u)\|_{(H_0^{1,p}(X, m))^*} \leq c_6(1 + \|u\|_{H_0^{1,p}(X, m)}^{p-1}),$$

$u \in H_0^{1,p}(X, m)$ , with a constant  $c_6 > 0$ .

Similarly as in the case of (30) we may invoke a general solution theory [28, 33], provided some regularity conditions are satisfied. In addition to (40) we will require the versions

$$(41) \quad \langle a(\partial f), \partial f \rangle \geq c_1 \|f\|_{1,p}^p - c_2 \|f\|_{L_2(X, m)}^2 \quad \text{for all } f \in H_0^{1,p}(X, m)$$

with constants  $c_1 > 0$ ,  $c_2 \geq 0$  and

$$(42) \quad \langle a(\partial f) - a(\partial g), \partial f - \partial g \rangle \geq c_3 \|f - g\|_{L_2(X, m)}^2 \quad \text{for all } f, g \in H_0^{1,p}(X, m),$$

with  $c_3 > 0$  of (33) and (35) with the left hand sides interpreted in the sense of duality. Finally, assume that for all  $u, v, w$  from the image  $Im \partial_p$  of  $H_0^{1,p}(X, m)$  under  $\partial_p$ ,

$$(43) \quad \text{the function } \lambda \mapsto \langle a(u + \lambda v), w \rangle \text{ is continuous at zero.}$$

*Remark 5.3.* Note that if (40) is valid and  $a = (a_x)_x$  is decomposable as before, the relation

$$\lim_{\lambda \rightarrow 0} \langle a_x(u(x) + \lambda v(x)), z(x) \rangle_{\mathcal{H}_x} = \langle a_x(v(x)), z(x) \rangle_{\mathcal{H}_x}$$

for  $m$ -a.e.  $x \in X$  implies (43), because

$$| \langle a(\partial f + \lambda \partial g), \partial h \rangle | \leq c(1 + \|f\|_{1,p} + \|g\|_{1,p}) \|h\|_{1,p}, \quad f, g, h \in H_0^{1,p}(X, m),$$

as one can easily verify.

A continuous  $(\mathcal{F}_t)$ -adapted process  $u = (u(t))_{t \in [0, T]}$  is called *a solution to (39)* if

$$\mathbb{E} \int_0^T (\|u(t)\|_{1,p}^p + \|u(t)\|_{L_2(X, m)}^2) dt < \infty$$

and

$$u(t) = u(0) + \int_0^t \partial^* a(\partial \tilde{u}(s)) ds + \int_0^t \sqrt{Q} dW(s), \quad t \in [0, T],$$

seen as an identity of  $(H_0^{1,p}(X, m))^*$ -valued functions, where  $\tilde{u}$  is any  $H_0^{1,p}(X, m)$ -valued progressively measurable  $dt \otimes d\mathbb{P}$ -version of  $u$ .

The following is a special case of the classical results in [28, 35].

**Theorem 5.3.** *Let  $2 \leq p < \infty$ ,  $m(X) < \infty$  and assume that  $a$  satisfies (40), (41), (42) and (43). Let*

$$\mathbb{E} \int_X u_0^2(x) m(dx) < \infty.$$

*Then (39) has a unique solution  $u$ .*

*Examples 5.1.* A specific example is given by the following *stochastic  $p$ -Laplace equation*: Let  $a = (a_x)_{x \in X}$  with

$$a_x(v(x)) := \|v(x)\|_{\mathcal{H}_x}^{p-2} v(x), \quad v \in Im \partial_p.$$

We have  $\|a(v)\|_{L_q(X, m, (\mathcal{H}_x)_x)} = \|v\|_{L_p(X, m, (\mathcal{H}_x)_x)}^{p-1}$  by Hölder's inequality, hence (40) holds. Condition (43) rewrites

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \int_X (\|\partial(u + \lambda v)(x)\|_{\mathcal{H}_x}^{p-2} \langle \partial(u + \lambda v)(x), \partial w(x) \rangle_{\mathcal{H}_x} \\ - \|\partial u(x)\|_{\mathcal{H}_x}^{p-2} \langle \partial u(x), \partial w(x) \rangle_{\mathcal{H}_x}) m(dx) = 0. \end{aligned}$$

But this follows by dominated convergence, the pointwise limit being obvious from the continuity of  $\|\cdot\|_{\mathcal{H}_x}$  for fixed  $x$  and a dominating integrable function being provided by

$$c (\|\partial v(x)\|_{\mathcal{H}_x}^{p-1} + \|\partial v(x)\|_{\mathcal{H}_x}^{p-1}) \|\partial w(x)\|_{\mathcal{H}_x}^{p-1}.$$

(42) holds with  $c_4 = 0$  because

$$\begin{aligned} & \int_X (\|\partial f(x)\|_{\mathcal{H}_x}^p + \|\partial g(x)\|_{\mathcal{H}_x}^p - \|\partial f(x)\|_{\mathcal{H}_x}^{p-2} \langle \partial f(x), \partial g(x) \rangle_{\mathcal{H}_x} \\ & \quad - \|\partial g(x)\|_{\mathcal{H}_x}^{p-2} \langle \partial f(x), \partial g(x) \rangle_{\mathcal{H}_x}) m(dx) \\ & \geq \int_X (\|\partial f(x)\|_{\mathcal{H}_x}^{p-1} - \|\partial g(x)\|_{\mathcal{H}_x}^{p-1}) (\|\partial f(x)\|_{\mathcal{H}_x} - \|\partial g(x)\|_{\mathcal{H}_x}) \geq 0. \end{aligned}$$

Condition (41) follows immediately if we assume the validity of a *p-Poincaré inequality*

$$(44) \quad \|f\|_{L_p(X,m)}^p \leq c_P \int_X \|\partial f(x)\|_{\mathcal{H}_x}^p m(dx)$$

with some  $c_P > 0$  for all  $f \in L_p(X, m)$  with  $\int_X f dm = 0$ . For smooth bounded Euclidean domains (44) follows by classical arguments. A non-classical case we are particularly interested in arises if  $(\mathcal{E}, \mathcal{F})$  is a regular resistance form [25, 26], with a Dirichlet domain  $\mathcal{F}$  in the sense that there is a point  $p \in X$  such that  $f(p) = 0$  for all  $f \in \mathcal{F}$ . Note that as elements of  $\mathcal{F}$  are continuous in the resistance metric, pointwise evaluation makes sense. Examples include regular harmonic structures on p.c.f. self-similar sets, [25], but also non self-similar spaces [47] and some infinitely ramified sets such as Sierpinski carpets [2]. In this case we have

$$(45) \quad \sup_x |f(x)| \leq c \mathcal{E}(f)^{1/2}, \quad x \in X,$$

with some  $c > 0$  and for all  $f \in \mathcal{F}$ , see for instance [25, Lemma 5.2.8] and its proof. Under the previously made assumptions  $m(X) < \infty$  and  $2 \leq p < \infty$  inequality (44) now follows immediately from (45). Another situation where cases of (44) are easily verified arises if a regular Dirichlet form admits a *Sobolev inequality*

$$\|f\|_{2d_s/(d_s-2)} \leq c \mathcal{E}(f)^{1/2}$$

for all  $f \in \mathcal{F}$  with  $\int_X f dm = 0$ , where  $d_s > 2$  is the spectral dimension. Together with  $m(X) < \infty$  this implies (44) for  $2 \leq p \leq 2d_s/(d_s - 2)$ .

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